Formal Power Series

The convert/FormalPowerSeries functionality was completely rewritten for Maple 2022. It offers a number of advantages over previous versions:

- Closed-form solutions can be found in a number of cases where previous versions failed.
- Solutions in terms of m-fold hypergeometric sequences for arbitrary positive integers m are now supported in more cases than before.
- Notwithstanding the name, formal Laurent and Puiseux series (i.e., with negative or fractional exponents) can be computed as well, now in more cases than before.
- convert/FormalPowerSeries will automatically attempt to return the series coefficients in purely real form, making the previous option *makereal* obsolete.
- In a number of cases, the new code returns more compact answers than previous versions.
- If a closed form expression for the power series coefficients cannot be found, and a recurrence relation of degree 1 or 2 exists, it will be returned instead. Previously, only linear recurrences could be computed, and would only be returned if option *recurrence* was specified.
- When a recurrence relation is returned, now the initial conditions are given as well.
- Additional options give more control over the underlying algorithm(s) used and the form of the output.

Maple 2021	**Maple 2022**
More closed-form solutions, notably, for sums of several terms and Puiseux solutions.	

$$convert\left(\left(-\frac{z}{2}+\frac{z^3}{6}\right)\arctan(z), FormalPowerSeries\right)$$

$$\left(-\frac{1}{2}\,z+\frac{1}{6}\,z^3\right)\arctan(z) \qquad (1)$$

$$map\left(convert, expand\left(\left(-\frac{z}{2}+\frac{z^3}{6}\right)\arctan(z)\right), FormalPowerSeries\right)$$

$$\sum_{k=0}^{\infty}\left(-\frac{(-1)^k z^{2k+2}}{4k+2}\right)+\left(\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+4}}{12k+6}\right) \qquad (2)$$

$$convert\left(\left(-\frac{z}{2}+\frac{z^3}{6}\right)\arctan(z), FormalPowerSeries\right)$$

$$-\frac{z^2}{6}+\frac{5}{9}+\left(\sum_{n=0}^{\infty}\frac{(4n-5)\,(-1)^n z^{2n}}{3\,(2n-1)\,(2n-3)}\right) \qquad (3)$$

$$convert(\arctan(z)+\arcsin(z), FormalPowerSeries)$$

$$\arctan(z)+\arcsin(z) \qquad (4)$$

$$map(convert, \arctan(z)+\arcsin(z), FormalPowerSeries)$$

$$\left(\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{2k+1}\right)+\left(\sum_{k=0}^{\infty}\frac{(2k)!\,4^{-k} z^{2k+1}}{k!^2\,(2k+1)}\right) \qquad (5)$$

$$convert(\arctan(z)+\arcsin(z), FormalPowerSeries)$$

$$\sum_{n=0}^{\infty}\frac{\left((-1)^n n!^2+(2n)!\,4^{-n}\right) z^{2n+1}}{(2n+1)\,n!^2} \qquad (6)$$

$$convert(\arctan(z)+\arcsin(z), FormalPowerSeries, output=expanded)$$

$$\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{2n+1}\right)+\left(\sum_{n=0}^{\infty}\frac{(2n)!\,4^{-n} z^{2n+1}}{(2n+1)\,n!^2}\right) \qquad (7)$$

$$convert\left(\frac{\left(1-\sqrt{1-4z}\right)^2 z^2}{4\sqrt{1-4z}}, FormalPowerSeries\right)$$

$$\frac{\left(1-\sqrt{1-4z}\right)^2 z^2}{4\sqrt{1-4z}} \qquad (8)$$

$$convert\left(\frac{\left(1-\sqrt{1-4z}\right)^2 z^2}{4\sqrt{1-4z}}, FormalPowerSeries\right)$$

$$\sum_{n=0}^{\infty}\frac{(2n+2)!\,(n+2)\,(n+1)\,z^{n+4}}{(n+2)!^2} \qquad (9)$$

$convert\left(\sqrt{\sqrt{8 z^3+1}-1}, FormalPowerSeries\right)$ $\sqrt{\sqrt{8 z^3+1}-1}$ **(10)**	$convert\left(\sqrt{\sqrt{8 z^3+1}-1}, FormalPowerSeries\right)$ $\sum_{n=0}^{\infty} \frac{1}{(2 n+1)!^2}\left((-1)^n 2^{-n+1} (2 n+1) (4 n)! z^{3 n+\frac{3}{2}}\right)$ **(11)**
Solutions in purely real form by default.	
$convert(\sin(z)+z\cos(z), FormalPowerSeries)$ $\sum_{k=0}^{\infty}\left(-\frac{\mathrm{I}\,\mathrm{I}^k (k+1)}{2 k!}+\frac{\mathrm{I}\,(-\mathrm{I})^k (k+1)}{2 k!}\right) z^k$ **(12)** $convert(\sin(z)+z\cos(z), FormalPowerSeries, makereal)$ $\sum_{k=0}^{\infty}\frac{\sin\left(\frac{k\pi}{2}\right)(k+1) z^k}{k!}$ **(13)**	$convert(\sin(z)+z\cos(z), FormalPowerSeries)$ $\sum_{n=0}^{\infty}\frac{2(-1)^n (n+1) z^{2 n+1}}{(2 n+1)!}$ **(14)**
$convert\left(\ln\left(1+z+z^2+z^3\right), FormalPowerSeries\right)$ $\sum_{k=0}^{\infty}\left(-\frac{(-1)^{k+1}}{k+1}-\frac{\mathrm{I}^{k+1}}{k+1}-\frac{(-\mathrm{I})^{k+1}}{k+1}\right) z^{k+1}$ **(15)** $convert\left(\ln\left(1+z+z^2+z^3\right), FormalPowerSeries, makereal\right)$ $\sum_{k=0}^{\infty}\frac{\left((-1)^k+2\sin\left(\frac{k\pi}{2}\right)\right) z^{k+1}}{k+1}$ **(16)**	$convert\left(\ln\left(1+z+z^2+z^3\right), FormalPowerSeries\right)$ $\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1}}{n+1}\right)+\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^{2 n+2}}{n+1}\right)$ **(17)**

$$convert\left(\frac{-z^6+3z^2}{-3z^4+1}, FormalPowerSeries\right)$$

$$\frac{z^2}{3}+\left(\sum_{k=0}^{\infty}\left(2\sqrt{3}\left(\left(-\frac{I}{3}3^{3/4}\right)^k\left(-\frac{3^{3/4}}{3}\right)^k\left(\frac{I}{3}3^{3/4}\right)^k-\left(-\frac{I}{3}3^{3/4}\right)^k\left(-\frac{3^{3/4}}{3}\right)^k\left(\frac{3^{3/4}}{3}\right)^k+\left(-\frac{I}{3}3^{3/4}\right)^k\left(\frac{I}{3}3^{3/4}\right)^k\left(\frac{3^{3/4}}{3}\right)^k-\left(-\frac{3^{3/4}}{3}\right)^k\left(\frac{I}{3}3^{3/4}\right)^k\left(\frac{3^{3/4}}{3}\right)^k\right)z^k\right)\bigg/\left(9\left(-\frac{I}{3}3^{3/4}\right)^k\left(-\frac{3^{3/4}}{3}\right)^k\left(\frac{I}{3}3^{3/4}\right)^k\left(\frac{3^{3/4}}{3}\right)^k\right)\right) \quad \textbf{(18)}$$

$$convert\left(\frac{-z^6+3z^2}{-3z^4+1}, FormalPowerSeries, makereal\right)$$

$$\frac{z^2}{3}+\sum_{k=0}^{\infty}\left(-\frac{1}{9}\left(2z^k3^{\frac{k}{4}+\frac{1}{2}}\left(2\cos\left(\frac{k\pi}{2}\right)-(-1)^k-1\right)\right)\right) \quad \textbf{(19)}$$

$$convert\left(\frac{-z^6+3z^2}{-3z^4+1}, FormalPowerSeries\right)$$

$$\frac{z^2}{3}+\left(\sum_{n=0}^{\infty}8\,3^{n-1}z^{4n+2}\right) \quad \textbf{(20)}$$

More compact answers.

$$convert\left(\frac{1}{(q_1 - z^2)\cdot(q_2 - z^3)}, FormalPowerSeries, z\right)$$

$$\sum_{k=0}^{\infty} ((-2 \, (((-1)^{2/3} q_2^{1/3})^k (\sqrt{q_1})^k ($$
$$-\sqrt{q_1})^k (q_2^{1/3})^k (-1)^{2/3} q_1^3$$
$$+ 2 \, ($$
$$-(-1)^{1/3} q_2^{1/3})^k (\sqrt{q_1})^k ($$
$$-\sqrt{q_1})^k (q_2^{1/3})^k q_2^{4/3} (-1)^{2/3} q_1$$
$$- ((-1)^{2/3} q_2^{1/3})^k ($$
$$-(-1)^{1/3} q_2^{1/3})$$
$$^k (\sqrt{q_1})^k (q_2^{1/3})^k (-1)^{2/3} q_1^{3/2} q_2$$
$$+ ((-1)^{2/3} q_2^{1/3})^k ($$
$$-(-1)^{1/3} q_2^{1/3})^k ($$
$$-\sqrt{q_1})^k (q_2^{1/3})^k (-1)^{2/3} q_1^{3/2} q_2$$
$$+ 2 \, ((-1)^{2/3} q_2^{1/3})^k (\sqrt{q_1})^k ($$
$$-\sqrt{q_1})^k (q_2^{1/3})^k (-1)^{1/3} q_1^3$$
$$+ ((-1)^{2/3} q_2^{1/3})^k ($$
$$-(-1)^{1/3} q_2^{1/3})$$
$$^k (\sqrt{q_1})^k (q_2^{1/3})^k (-1)^{1/3} q_1^{3/2} q_2$$
$$- ((-1)^{2/3} q_2^{1/3})^k ($$
$$-(-1)^{1/3} q_2^{1/3})^k ($$
$$-\sqrt{q_1})^k (q_2^{1/3})^k (-1)^{1/3} q_1^{3/2} q_2$$
$$- 2 \, ($$

(21)

$$convert\left(\frac{1}{(q_1 - z^2)\cdot(q_2 - z^3)}, FormalPowerSeries, z\right)$$

$$\left(\sum_{n=0}^{\infty} \left(\left(-q_1^{2-\frac{n}{2}} + (-1)^n q_1^{2-\frac{n}{2}} + 2 q_2^{-\frac{1}{3}-\frac{n}{3}} q_1^{5/2} - 2 q_1^{-\frac{n}{2}+\frac{1}{2}} q_2 - (-1)^n q_1^{-1-\frac{n}{2}} q_2^2 + 2 q_2^{\frac{2}{3}-\frac{n}{3}} q_1 - q_1^{-1-\frac{n}{2}} q_2^2 \right) z^n \right) \Big/ \left(2 \left(q_1^3 - q_2^2\right)\left(q_1^{3/2} + q_2\right)\right)\right) + \left(\sum_{n=0}^{\infty} \frac{q_1 q_2^{-n-1} z^{3n}}{q_2^{4/3} + q_2^{2/3} q_1 + q_1^2}\right) + \sum_{n=0}^{\infty} \left(-\frac{q_2^{-n-\frac{2}{3}} z^{3n+1}}{q_2^{4/3} + q_2^{2/3} q_1 + q_1^2} \right)$$

(22)

Recurrence relations returned automatically if no closed form can be found, with initial conditions. Non-linear (degree 2) recurrences can be computed.	
$convert(\arcsin(z)^3, FormalPowerSeries)$ $$\arcsin(z)^3 \quad (23)$$ $convert(\arcsin(z)^3, FormalPowerSeries, recurrence)$ $$k^4\, a(k) - 2\,(k+1)\,(k+2)\,(k^2+2\,k+2)\,a(k+2) + (k+1)\,(k+2)\,(k+3)\,(k+4)\,a(k+4) = 0 \quad (24)$$	$convert(\arcsin(z)^3, FormalPowerSeries)$ $$\sum_{n=0}^{\infty} A(n)\, z^{n+1}, RESol(\{(n^4+4\,n^3+6\,n^2+4\,n+1)\,A(n) + (-2\,n^4-18\,n^3-62\,n^2-98\,n-60)\,A(n+2) + (n^4+14\,n^3+71\,n^2+154\,n+120)\,A(n+4) = 0\}, \{A(n)\}, \{A(0)=0, A(1)=0, A(2)=1, A(3)=0\}, INFO) \quad (25)$$
$convert\left(\frac{z}{e^z-1}, FormalPowerSeries\right)$ $$\frac{z}{e^z-1} \quad (26)$$ $convert\left(\frac{z}{e^z-1}, FormalPowerSeries, recurrence\right)$ $$\frac{z}{e^z-1} \quad (27)$$	$convert\left(\frac{z}{e^z-1}, FormalPowerSeries\right)$ $$\sum_{n=0}^{\infty} A(n)\, z^n, RESol\left(A(n+3) + \frac{1}{n+4}\left(A(n+2) + \left(\sum_{\_k=1}^{n+2} A(\_k)\, A(n+3-\_k)\right)\right), \{A(n)\}, \left\{A(0)=1, A(1)=-\frac{1}{2}, A(2)=\frac{1}{12}\right\}, INFO\right) \quad (28)$$

$convert(\mathrm{LambertW}(z), FormalPowerSeries)$ $$\mathrm{LambertW}(z) \quad (29)$$ $convert(\mathrm{LambertW}(z), FormalPowerSeries, recurrence)$ $$\mathrm{LambertW}(z) \quad (30)$$	$convert(\mathrm{LambertW}(z), FormalPowerSeries)$ $$\sum_{n=0}^{\infty} A(n)\, z^n, RESol\left(A(n+4) + \frac{1}{n+3}\left(A(n+3) + \left(\sum_{\_k=1}^{n+2} (\_k+1)\, A(\_k+1)\, A(n+3-\_k)\right)\right), \{A(n)\}, \left\{A(0)=0, A(1)=1, A(2)=-1, A(3)=\frac{3}{2}\right\}, INFO\right) \quad (31)$$
New method option (by default, all three methods are tried in sequence).	

	$convert\left(\arcsin(z)^2, FormalPowerSeries\right)$
	$$\sum_{n=0}^{\infty} \frac{2\, n!^2\, 4^n\, z^{2n+2}}{(2n+2)!} \quad (32)$$
	$convert\left(\arcsin(z)^2, FormalPowerSeries, method = hypergeometric\right)$
	$$\sum_{n=0}^{\infty} \frac{2\, n!^2\, 4^n\, z^{2n+2}}{(2n+2)!} \quad (33)$$
	$convert\left(\arcsin(z)^2, FormalPowerSeries, method = holonomic\right)$
	$$\sum_{n=0}^{\infty} A(n)\, z^{n+1}, RESol\left(\left\{\left(-n^2 - 2n - 1\right) A(n) + \left(n^2 + 5n + 6\right) A(n+2) = 0\right\}, \{A(n)\}, \{A(0) = 0, A(1) = 1\}, INFO\right) \quad (34)$$
	$convert\left(\arcsin(z)^2, FormalPowerSeries, method = quadratic\right)$
	$$\sum_{n=0}^{\infty} A(n)\, z^n, RESol\left(A(n+4) - \frac{1}{4(n+3)}\left(4(n+2) A(n+2) + \left(\sum_{\_k=2}^{n} (\_k+1) A(\_k+1) (n+3-\_k) A(n+3-\_k)\right) + \sum_{\_k=2}^{n+2} (-(\_k+1) A(\_k+1) (n+5-\_k) A(n+5-\_k))\right), \{A(n)\}, \{A(0) = 0, A(1) = 0, A(2) = 1, A(3) = 0\}, INFO\right) \quad (35)$$

	$convert(\tan(z), FormalPowerSeries)$ $\sum_{n=0}^{\infty} A(n)\, z^n, RESol\Bigg(A(n+3) + \frac{1}{(n+2)\,(n+3)}\Big(-2\,A(n+1) + \sum_{\_k=1}^{n} (-2\,(\_k+1)\,A(\_k+1)\,A(n+1-\_k))\Big), \{A(n)\}, \{A(0)=0, A(1)=1, A(2)=0\}, INFO\Bigg)$ **(36)** $convert(\tan(z), FormalPowerSeries, method = hypergeometric)$ $\tan(z)$ **(37)** $convert(\tan(z), FormalPowerSeries, method = holonomic)$ $\tan(z)$ **(38)** $convert(\tan(z), FormalPowerSeries, method = quadratic)$ $\sum_{n=0}^{\infty} A(n)\, z^n, RESol\Bigg(A(n+3) + \frac{1}{(n+2)\,(n+3)}\Big(-2\,A(n+1) + \sum_{\_k=1}^{n} (-2\,(\_k+1)\,A(\_k+1)\,A(n+1-\_k))\Big), \{A(n)\}, \{A(0)=0, A(1)=1, A(2)=0\}, INFO\Bigg)$ **(39)**
New output option (default: combined).	

	$convert\left((\sin(z)+\cos(z))^3, FormalPowerSeries\right)$
	$$\sum_{n=0}^{\infty}\left(-\frac{(-1)^n\,(9^n-3)\,z^{2n}}{2\,(2n)!}\right)+\left(\sum_{n=0}^{\infty}\frac{3\,(-1)^n\,(9^n+1)\,z^{2n+1}}{2\,(2n+1)!}\right) \quad \textbf{(40)}$$
	$convert\left((\sin(z)+\cos(z))^3, FormalPowerSeries, output=combined\right)$
	$$\sum_{n=0}^{\infty}\left(-\frac{(-1)^n\,(9^n-3)\,z^{2n}}{2\,(2n)!}\right)+\left(\sum_{n=0}^{\infty}\frac{3\,(-1)^n\,(9^n+1)\,z^{2n+1}}{2\,(2n+1)!}\right) \quad \textbf{(41)}$$
	$convert\left((\sin(z)+\cos(z))^3, FormalPowerSeries, output=expanded\right)$
	$$\sum_{n=0}^{\infty}\left(-\frac{(-1)^n\,9^n\,z^{2n}}{2\,(2n)!}\right)+\left(\sum_{n=0}^{\infty}\frac{3\,(-1)^n\,z^{2n}}{2\,(2n)!}\right)+\left(\sum_{n=0}^{\infty}\frac{3\,(-1)^n\,9^n\,z^{2n+1}}{2\,(2n+1)!}\right)+\left(\sum_{n=0}^{\infty}\frac{3\,(-1)^n\,z^{2n+1}}{2\,(2n+1)!}\right) \quad \textbf{(42)}$$